Chapter 2: Cardinality

Cummings' Exercises

Exercises with a $\star$ are considered "Notable Exercises" in the book.

Exercise 2.4. $\star$

This problem shows that "equinumerosity is an equivalence relation." (This justifies the notation $|A| = |B|$.) Let A, B and C be sets. For this problem only, we'll write $A \sim B$ to mean that A and B are equinumerous, meaning that there is a bijection from $A \to B$.

(a) Show that $A \sim A$.

(b) Show that if $A \sim B$ then $B \sim A$.

(c) Show that if $A \sim B$ and $B \sim C$, then $A \sim C$

Exercise 2.10. $\star$

Let S be the set of sequences (a_n) where, for each n, $a_n \in \{0, 1\}$. Is S countable or uncountable?

Exercise 2.12. $\star$

(a) Give an example of a collection of finitely many disjoint open intervals, or prove that this does not exist.

(b) Give an example of a collection of uncountably many disjoint open intervals, or prove that this does not exist.

Exercise 2.13. $\star$

Show that there are uncountably many irrational numbers.

Exercise 2.16. $\star$

Show that the smallest infinity is $|\mathbb{N}|$. That is, show that if $A \subseteq \mathbb{N}$, then either A is finite or $|A| = |\mathbb{N}|$.

Exercise 2.22. $\star$

A real number x is said to be *algebraic* (over the rationals) if it satisfies some polynomial equation (of positive degree)

$$a_n x^n + a_{n-1} x^{n-1} + a_{n-2} x^{n-2} + \ldots + a_1 x + a_0 = 0$$

where each $a_i \in \mathbb{Q}$. If a real number is not algebraic, then it is *transcendental.*

(a) Prove that there are countably many algebraic numbers. (You may use the fundamental theorem of algebra which says that a polynomial with degree n has at most n real roots.)

(b) Prove that there are uncountably many transcendental numbers.